SHILNIKOV PROBLEM IN FILIPPOV DYNAMICAL SYSTEMS

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ABSTRACT. In this paper we introduce the concept of a sliding Shilnikov orbit for 3D Filippov systems. Versions of the Shilnikov's Theorems are provided for those systems. Specifically, we show that arbitrarily close to a sliding Shilnikov orbit there exist countable infinitely many sliding periodic orbits, and for a particular system having this kind of connection we investigate the existence of continuous systems close to it having an ordinary Shilnikov homoclinic orbit. We also prove that, in general, a sliding Shilnikov orbit is a co-dimension 1 phenomena. Furthermore we provide a family $Z_{\alpha,\beta}$ of piecewise linear vector fields as a prototype of systems having a sliding Shilnikov orbit.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

The study of nonsmooth dynamical systems produces interesting and amazing mathematical challenges and plays an important part of so many applications in several branches of science (see, for instance, [15, 10, 19, 3] and the references therein). The present work focuses on the analysis of a typical phenomenon that occurs in this area which evidences a striking resemblance to Shilnikov homoclnic loop

Consider a smooth three dimensional vector field for which $p \in \mathbb{R}^3$ is a hyperbolic saddle–focus equilibrium admitting a two dimensional stable manifold and an one dimensional unstable manifold. In the classical theory of dynamical systems a *Shilnikov homoclinic orbit* Γ of this vector field is a trajectory connecting p to itself, bi–asymptotically. Under suitable genericity conditions this connection is a codimension one scenario, and its unfolding depends on the *saddle quantity* $\sigma = \lambda^u + \operatorname{Re}(\lambda^s_{1,2})$, where $\lambda^u > 0$, $\lambda^s_{1,2} \in \mathbb{C}$ $(\operatorname{Re}(\lambda^s_{1,2}) < 0)$ are the eigenvalues of p. In this case, when $\sigma > 0$ an intricate behaviour occurs when Γ is broken. Indeed, it is proved that there exists a compact hyperbolic invariant chaotic set $\mathcal{S}$ which contains countable infinitely many periodic orbits

of saddle type in any sufficiently small neighbourhood of Γ(see, for instance, [11, 12, 16, 17, 13, 18, 1]).

In the theory of nonsmooth dynamical systems the notion of solutions of a discontinuous differential equation is stated by the Filippov's convention (see [5]). In this context there exist some special points that must be distinguished and treated as typical singularities, one of those is a pseudo–equilibrium, which we shall introduce it formally later on this paper. This kind of singularity gives rise to the definition of the sliding homoclinic orbit, that is a trajectory, in the Filippov sense, connecting a pseudo–equilibrium to itself in an infinity time at least by one side (future or past). Particularly a *sliding Shilnikov orbit* is a sliding homoclinic orbit connecting a hyperbolic pseudo saddle–focus pseudo to it self.

The main goal of this paper is to produce versions of the Shilnikov's Theorems for systems having a sliding Shilnikov orbit. More especially, we show, in Subsection 1.2, that arbitrarily close to a sliding Shilnikov orbit there exist countable infinitely many sliding periodic orbits, and for a particular system having this kind of connection we investigate, in Section 3, the existence of continuous systems close to it having an ordinary Shilnikov homoclinic orbit. In Subsection 1.2, we also prove that, in general, a sliding Shilnikov orbit is a co-dimension 1 phenomena. Furthermore, in Section 2, we provide a family $Z_{\alpha,\beta}$ of discontinuous piecewise linear vector fields as a prototype of systems having a sliding Shilnikov orbit.

1.1. **Setting the problem.** In this subsection the basic theory of nonsmooth dynamical systems is given in order to define the sliding Shilnikov orbits and to state our main results.

Let U be an open bounded subset of $\mathbb{R}^3$. We denote by $\mathcal{C}^r(\overline{U}, \mathbb{R}^3)$ the set of all $\mathcal{C}^r$ vector fields $X : \overline{U} \to \mathbb{R}^3$ endowed with the topology induced by the norm $||X||_r = \sup\{||D^i X(x)|| : x \in \overline{U}, i \in \{0, 1, \ldots, r\}\}$. Here D^r is the identity operator for $r = 0$, and the rth–derivative for $r > 0$. In order to keep the uniqueness property of the trajectories of vector fields in $\mathcal{C}^0(\overline{U}, \mathbb{R}^3)$ we shall assume, additionally, that these vector fields are Lipschitz.

Given $h : K \to \mathbb{R}$ a differentiable function having 0 as a regular value we denote by $\Omega_h^r(K, \mathbb{R}^3)$ the space of piecewise vector fields Z in $\mathbb{R}^n$ such that

$$(1) \qquad\qquad Z(x) = \begin{cases} X(x), & \text{if} \quad h(x) > 0, \\ Y(x), & \text{if} \quad h(x) > 0, \end{cases}$$

with $X, Y \in \mathcal{C}^r(K, \mathbb{R}^3)$. As usual, system (1) is denoted by $Z = (X, Y)$ and the switching surface $h^{-1}(0)$ by Σ . So we are taking $\Omega_h^r(K, \mathbb{R}^3) = \mathcal{C}^r(K, \mathbb{R}^3) \times$

$\mathcal{C}^r(K, \mathbb{R}^3)$ endowed with the product topology. When the context is clear we shall refer the sets $\Omega_h^r(K, \mathbb{R}^3)$ and $\mathcal{C}^r(K, \mathbb{R}^3)$ only by Ω^r and $\mathcal{C}^r$, respectively.

The points on Σ where both vectors fields X and Y simultaneously point outward or inward from Σ define, respectively, the *escaping* Σ^e or *sliding* Σ^s regions, and the interior of its complement in Σ defines the *crossing region* Σ^c. The complementary of the union of those regions constitute by the *tangency* points between X or Y with Σ (see Figure 1).
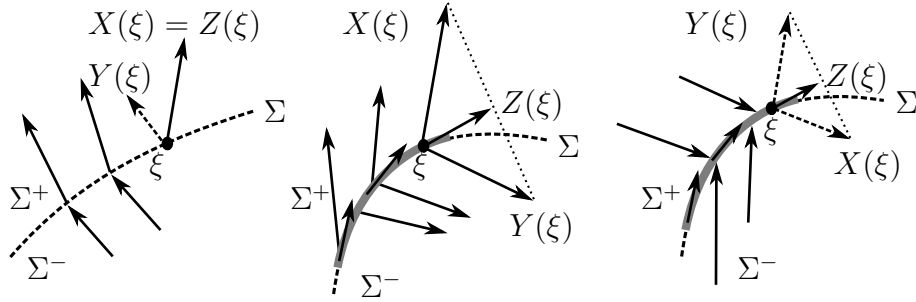

FIGURE 1. Definition of the vector field on Σ following Filippov's convention in the sewing, escaping, and sliding regions, respectively. This figure has been gotten from [2]

The points in Σ^c satisfy $Xh(\xi) \cdot Yh(\xi) > 0$, where Xh denote the derivative of the function h in the direction of the vector X, i.e. $Xh(\xi) = \langle \nabla h(\xi), X(\xi) \rangle$. The points in Σ^s (resp. Σ^e) satisfy $Xh(\xi) < 0$ and $Yh(\xi) > 0$ (resp. $Xh(\xi) > 0$ and $Yh(\xi) < 0$). Finally, the tangency points of X (resp. Y) satisfy $Xh(\xi) = 0$ (resp. $Yh(\xi) = 0$).

Now we define the *sliding vector field*

$$(2) \qquad \widetilde{Z}(\xi) = \frac{Yh(\xi)X(\xi) - Xh(\xi)Y(\xi)}{Yh(\xi) - Xh(\xi)}.$$

The local trajectory of the discontinuous piecewise differential system $\dot{x} = Z(x)$ passing through a point $p \in \mathbb{R}^3$ is given by the Filippov convention (see [5, 7]):

(i) for $p \in \mathbb{R}^3$ such that $h(p) > 0$ (resp. $h(p) < 0$) and taking the origin of time at p, the trajectory is defined as $\varphi_Z(t, p) = \varphi_X(t, p)$ (resp. $\varphi_Z(t, p) = \varphi_Y(t, p)$) for $t \in I_p$.

(ii) for $p \in \Sigma^c$ such that $(Xh)(p), (Yh)(p) > 0$ and taking the origin of time at p, the trajectory is defined as $\varphi_Z(t, p) = \varphi_Y(t, p)$ for $t \in I_p \cap \{t < 0\}$ and $\varphi_Z(t, p) = \varphi_X(t, p)$ for $t \in I_p \cap \{t > 0\}$. For the case $(Xh)(p), (Yh)(p) < 0$ the definition is the same reversing time;

(iii) for $p \in \Sigma^s$ and taking the origin of time at p, the trajectory is defined as $\varphi_Z(t,p) = \varphi_{\widetilde{Z}}(t,p)$ for $t \in I_p \cap \{t \geq 0\}$ and $\varphi_Z(t,p)$ is either $\varphi_X(t,p)$ or $\varphi_Y(t,p)$ or $\varphi_{\widetilde{Z}}(t,p)$ for $t \in I_p \cap \{t \leq 0\}$. For the case $p \in \Sigma^e$ the definition is the same reversing time;

(iv) For $p \in \partial\Sigma^c \cup \partial\Sigma^c \cup \partial\Sigma^c$ such that the definitions of trajectories for points in Σ in both sides of p can be extended to p and coincide, the orbit through p is this limiting orbit. We will call these points *regular tangency* points.

(v) for any other point (*singular tangency* points) $\varphi_Z(t,p) = p$ for all $t \in \mathbb{R}$;

Here φ_W denotes the flow of a vector field W.

Remark 1. *A tangency point $\xi \in \Sigma$ is called a visible fold of X (resp. Y) if $(X)^2 h(\xi) > 0$ (resp. $(Y)^2 h(\xi) < 0$). Analogously, reversing the inequalities, we define a invisible fold. Suppose that p is a visible fold of X such that $Y h(p) > 0$, then p is an example of a regular tangency point. In this case, taking the origin of time at p, the trajectory passing through p is defined as $\varphi_Z(t,p) = \varphi_1(t,p)$ for $t \in I_p \cap \{t \leq 0\}$ and $\varphi_Z(t,p) = \varphi_2(t,p)$ for $t \in I_p \cap \{t \geq 0\}$, where each φ_1, φ_2 is either φ_X or φ_Y or $\varphi_{\widetilde{Z}}$.*

A *pseudo–equilibrium* is a critical point $\xi^* \in \Sigma^{s,e}$ of the sliding vector field, i.e. $\widetilde{Z}(\xi^*) = 0$. When ξ^* is a hyperbolic critical point of $\widetilde{Z}$, it is called a *hyperbolic pseudo–equilibrium*. Particularly if $\xi^* \in \Sigma^s$ (resp. $\xi^* \in \Sigma^e$) is an unstable (resp. stable) hyperbolic focus of $\widetilde{Z}$ then we call ξ^* a *hyperbolic saddle–focus pseudo–equilibrium* or just *hyperbolic pesudo saddle–focus*.

In order to study the orbits of the sliding vector field it is convenient to define the ($\mathcal{C}^r$) normalized sliding vector field

$$(3) \qquad \widehat{Z}(\xi) = (Y h(\xi) - X h(\xi))\widetilde{Z}(\xi) = Y h(\xi)X(\xi) - X h(\xi)Y(\xi),$$

which has the same phase portrait of $\widetilde{Z}$ reversing the direction of the flow in the escaping region. Indeed, system (3) is obtained from (2) through a time rescaling multiplying (2) by the function $Y h(\xi) - X h(\xi)$ which is positive (resp. negative) for $\xi \in \Sigma^s$ (resp. $\xi \in \Sigma^e$).

Definition 1. *Let $Z = (X,Y)$ be a piecewise continuous vector field having a hyperbolic pseudo saddle–focus $p \in \Sigma^s$ (resp. $p \in \Sigma^e$). We assume that there exists a tangential point $q \in \partial\Sigma^s$ (resp. $q \in \partial\Sigma^e$) which is a visible fold point of the vector field X such that*

(j) *the orbit passing through q following the sliding vector field $\widetilde{Z}$ converges to p backward in time (resp. forward in time);*

(jj) *the orbit starting at q and following the vector field X spends a time $t_0 > 0$ (resp. $t_0 < 0$) to reach p.*

So through p and q a sliding loop Γ *is easily characterized. We call* Γ *a sliding Shilnikov orbit (see Figures 2 for* $\alpha = 0$, *3, and 5).*

Remark 2. *Given* $Z = (X, Y) \in \Omega^r$ *it is worth to say that if* $p \in \partial\Sigma^{e,s}$ *is a fold–regular point of* Z, *that is* p *is a fold of* X *(resp. of* Y) *such that* $Yh(p) \neq 0$ *(resp.* $Xh(p) \neq 0$*), then the sliding vector field* $\widetilde{Z}$ *(2) is transverse to* $\partial\Sigma$ *at* p. *A proof of this fact can be found in* [15]

1.2. **Main results.** In the theory of ordinary differential equations a Shilnikov homoclinic orbit of a 3D vector field is a co–dimension 1 phenomenon in $\mathcal{C}^r$. Our first main result shows that the sliding Shilnikov is also a co-dimension 1 phenomenon in Ω^r.
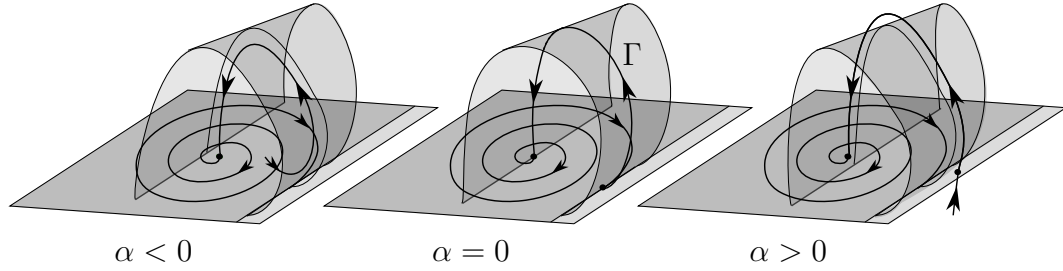

$\alpha < 0$ $\alpha = 0$ $\alpha > 0$

FIGURE 2. Unfolding $Z_\alpha = (X_\alpha, Y_\alpha)$ of a sliding Shilnikov orbit Γ in $Z_0 = (X_0, Y_0) \in \Omega^r$.

Theorem A. *Assume that* $Z_0 = (X_0, Y_0) \in \Omega^r$ *(with* $r \geq 1$*) has a sliding Shilnikov orbit* Γ_0 *and let* $W \subset \Omega^r$ *be a small neighbourhood of* Z_0. *Then there exists a* $\mathcal{C}^1$ *function* $g : W \to \mathbb{R}$ *having 0 as a regular value such that* $Z \in W$ *has a sliding Shilnikov orbit* Γ *if and only if* $g(Z) = 0$.

Proof. For simplicity we assume that $h(x, y, z) = z$, that is $\Sigma = \{z = 0\}$. Let $Z_0 = (X_0, Y_0) \in \Omega^r$ having a sliding Silnikov orbit Γ_0. We assume that Γ_0 is a sliding loop through a pseudo–equilibrium of a focus–saddle type $p_0 \in \Sigma^s$ and a tangential point q_0 which is a visible fold point for the vector field X_0. The case when $p_0 \in \Sigma^e$ would follows similarly.

Let $\gamma_0 = B_r(q_0) \cap \partial\Sigma^s$. Here $B_a(q_0) \subset \Sigma$ is the planar ball with center at q_0 and radius r. Of course γ_0 is a branch of the fold line contained in the boundary of the sliding region $\partial\Sigma^s$. We remark that in the sliding region the orbit of the sliding vector field is always transversal to the fold line. In addition, the orbits of $\widetilde{Z}_0$ through the points of γ_0 converge to p_0 in backward time. The forward saturation of γ_0 through the flow of X_0 meets Σ in a curve μ_0 in a finite time. Moreover $p_0 \in \mu_0$.

Let W be a small neighborhood of $Z_0 \in \Omega^r$. So associated to each $Z \in W$ we can define similar objects: p_Z, γ_Z and μ_Z. Clearly Z will have a sliding Shilnikov orbit if and only if $p_Z \in \mu_Z \neq$.

We may assume that, in suitable local coordinate system (x, y) around $p_0 = (0,0) \in \Sigma^s$, μ_0 is the graph of a function $y = r(x)$. So for $Z \in W$, μ_Z is given by $y = k_Z(x) = a_0 + a_1 x + \mathcal{O}_2(x)$ with a_0, a_1 small parameters.

Let $p_Z = (x^*, y^*)$ and define $g : W \to \mathbb{R}$ by $g(Z) = k_Z(x^*) - y^*$. Of course g is a $\mathcal{C}^1$ function and $g(Z_0) = 0$. We prove now that 0 is a regular value of g, that is $g'(Z_0) \neq 0$.

First of all we note that $g(Z) = 0$ if and only if $p_Z \in \mu_Z$ (sliding Shilnikov orbit). Since

$$g'(Z_0) = \lim_{v \to 0} \frac{g(Z_v) - g(Z_0)}{v}$$

for any curve $Z_v \in \Omega^r$ converging to Z_0, we can take Z_v in such a way that $p_Z = (0,0)$ and $k_Z(x) = v$ (constant). Hence $g(Z_v) = v$ and so $g'(Z_0) = 1$. It concludes the proof of this lemma.   □

Our second main result is a version of Shilnikov's theorem for sliding Shilnikov orbits.

Theorem B. *Assume that $Z_0 = (X_0, Y_0) \in \Omega^r$ (with $r \geq 0$) has a sliding Shilnikov orbit Γ_0 and let $Z_\alpha = (X_\alpha, Y_\alpha) \in \Omega^r$ be an unfolding of Z_0 with respect to Γ_0. Then the following statements hold:*

  (a) for $\alpha = 0$ every neighbourhood $G \subset \mathbb{R}^3$ of Γ_0 contains countable infinitely many sliding periodic orbits of Z_0;
  (b) for every $|\alpha| \neq 0$ sufficiently small there exists a neighbourhood $G_\alpha \subset \mathbb{R}^3$ of Γ_0 containing a finite number $N(\alpha) > 0$ of sliding periodic orbits of Z_α. Moreover $N(\alpha) \to \infty$ when $\alpha \to 0$;
  (c) for every neighbourhood $G \subset \mathbb{R}^3$ of Γ_0 there exists $|\alpha_0| \neq 0$ sufficiently small such that G contains a finite number $N_G(\alpha_0) > 0$ of sliding periodic orbits of Z_α. Moreover $N_G(\alpha) \to \infty$ when $\alpha \to 0$.

Proof. We assume that Γ_0 is a loop through $p_0 \in \Sigma^s$ and $q_0 \in \partial\Sigma^s$. The case $p_0 \in \Sigma^e$ and $q_0 \in \partial\Sigma^e$ would follow analogously.

To prove statement (a) let $\gamma_r = B_r(q_0) \cap \partial\Sigma^s$ and let S_r be the backward saturation of γ_r through the flow of the sliding vector field $\widetilde{Z}$. The forward saturation of γ_r through the flow of X meets Σ in a curve μ_r in a finite time. So

$$S_r \cap \mu_r = \bigcup_{i=1}^{\infty} I_i,$$

where $I_i \cap I_j = \emptyset$ if $i \neq j$. The sequence of sets $(I_i)_{i=1}^{\infty}$ can be taken such that $I_i \to \{p_0\}$ (see Figure 3).

For each $i = 1, 2, \ldots$, we define J_i as the intersection between the backward saturation of I_i through the flow of X with the curve γ_r. Clearly $J_i \cap J_j = \emptyset$ if $i \neq j$ and $J_i \to \{q_0\}$.
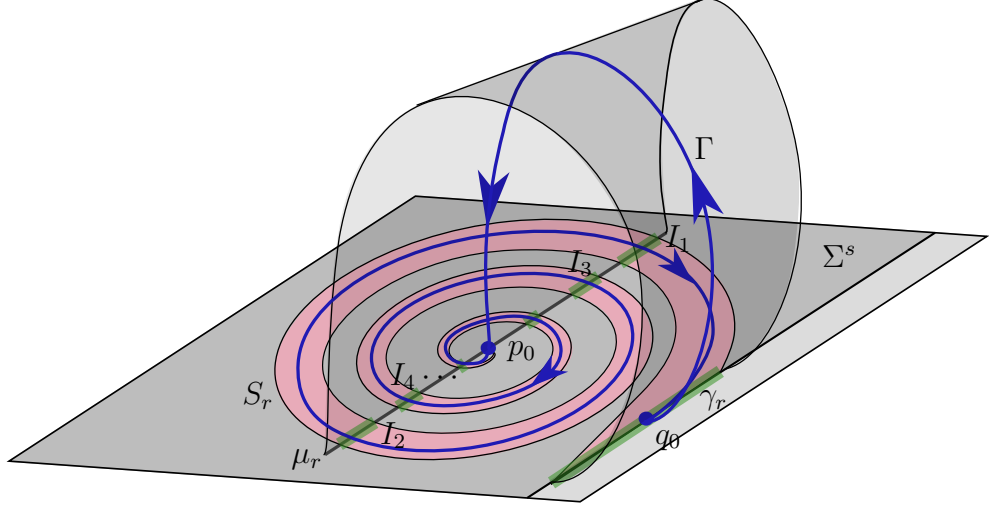
FIGURE 3. A schematic representation of a Shilnikov sliding orbit Γ.

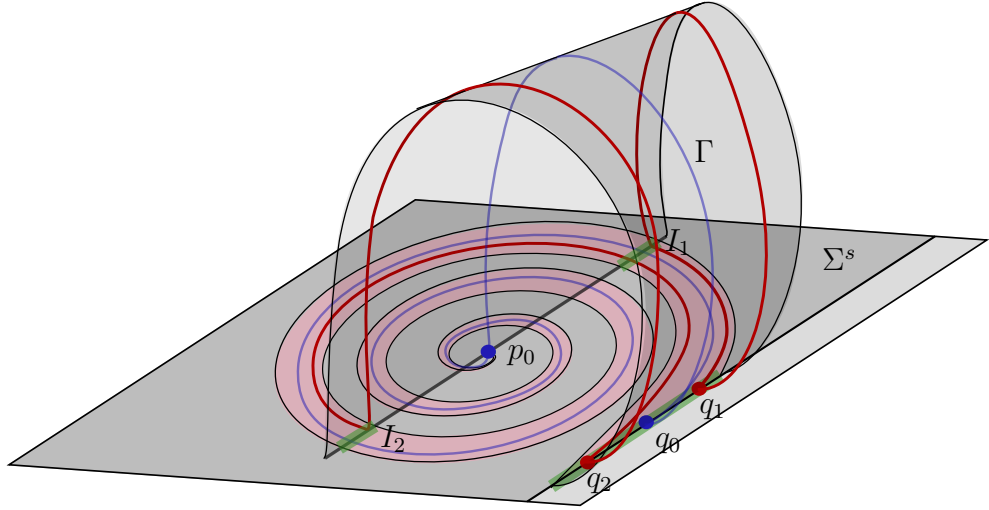

FIGURE 4. Periodic orbits close to a Shilnikov sliding orbit Γ.

For $\xi \in \Sigma^s$ and $z \in \mathbb{R}^3$ let $\varphi^s(t, \xi)$ and $\varphi^X(t, z)$ be the flows of the sliding vector field $\widetilde{Z}$ and X, respectively.

In what follows we define the applications $\psi_i : \gamma_r \to J_i$. For $\xi \in \gamma_r$ there exists $t_i^s(\xi) < 0$ such that $\xi_i(\xi) = \varphi^s(t_i^s(\xi), \xi) \in I_i$; and there exists $t_i^X(\xi) < 0$

such that $f^X(t_i^X(\xi), \xi_i(\xi)) \in J_i$. So we take $\psi_i(\xi) = f^X(t_i^X(\xi), \xi_i(\xi))$. Note that ψ_i is a composition of $\mathcal{C}^r$ function, being then itself a $\mathcal{C}^r$ function.

It is easy to see that each fixed point of ψ_i corresponds to a sliding periodic orbit of Z (see Figure 4). Since for each $i = 1, 2, \ldots, \psi_i$ is a contraction. So we obtain a sequence $(q_i)_{i=1}^{\infty}$ such that $q_i \in J_i$ and $\psi_i(q_i) = q_i$. Hence we conclude that there exists a sequence of sliding periodic orbits of Z passing through q_i. The proof follows just by observing that $q_i \to q_0$.

In what follows we prove the statements (b) and (c). Firstly for $|\alpha| \neq 0$ sufficiently small we build elements γ_r^{α}, S_r^{α} and μ_r^{α} similarly to the elements γ_r, S_r and μ_r, respectively.

Since the new pseudo–equilibrium p_{α} is not in μ_r^{α}, the intersection $S_r^{\alpha} \cap \mu_r^{\alpha}$ has only a finite number $N(\alpha)$ of disjoint sets I_i. Furthermore the number of disjoint sets $N(\alpha)$ in this intersection goes to infinity when α goes to 0, and they converges to $\{p\}$ when $i \to \infty$. From here the proof of statement (b) follows analogously to the proof of statement (a).

For a fixed neighbourhood G of Γ_0 there exists $|\alpha| \neq 0$ sufficiently small such that $p_{\alpha} \in G \cap \Sigma$, because $p_{\alpha} \to 0$ when $\alpha \to 0$, so that $\mu_{\alpha} \subset G \cap \Sigma$. From here the proof of statement (c) follows analogously to the proof of statement (b). $\qquad\square$

2. A PIECEWISE LINEAR MODEL

In this section we present a 2–parameter family of discontinuous piecewise linear dynamical system $Z_{\alpha,\beta}$ admitting a sliding Shilnikov orbit $\Gamma_{\alpha,\beta}$.

For $\alpha > 0$ and $\beta > 0$, consider the following discontinuous piecewise linear vector field.

$$(4) \qquad Z_{\alpha,\beta}(x,y,z) = \begin{cases} X_{\alpha,\beta}(x,y,z) = \begin{pmatrix} -\alpha \\ x - \beta \\ y - \dfrac{3\beta^2}{8\alpha} \end{pmatrix} & \text{if} \quad z > 0, \\[2em] Y_{\alpha,\beta}(x,y,z) = \begin{pmatrix} \alpha \\ \dfrac{3\alpha}{\beta}y + \beta \\ \dfrac{3\beta^2}{8\alpha} \end{pmatrix} & \text{if} \quad z < 0. \end{cases}$$

The plane $\Sigma = \{z = 0\}$ is a switching manifold for system (4), which can be decomposed as $\Sigma = \overline{\Sigma^c} \cup \overline{\Sigma^s} \cup \overline{\Sigma^e}$ being

$$\Sigma^c = \left\{ (x,y,0) : y > \frac{3\beta^2}{8\alpha} \right\}, \quad \Sigma^s = \left\{ (x,y,0) : y < \frac{3\beta^2}{8\alpha} \right\} \quad \text{and} \quad \Sigma^e = \emptyset.$$

Thus $p = (0,0,0) \in \Sigma^s$. Moreover $c = (\beta, 3\beta^2/(8\alpha), 0)$ is a cuspid-regular singularity for system (4) (see Figure 5).

Proposition 3. *For every positive real numbers α and β the following statements hold:*

(a) *the origin $p = (0,0,0)$ is a hyperbolic pseudo saddle–focus of system $Z_{\alpha,\beta}$ (4) in such way that its projection onto Σ is an unstable hyperbolic focus of the sliding vector field $\widetilde{Z}_{\alpha,\beta}$ (2) associated with (4);*

(b) *there exists a sliding Shilnikov orbit $\Gamma_{\alpha,\beta}$, connecting $p = (0,0,0)$ to itself, passing through the fold–regular point $q = \left(3\beta/2, 3\beta^2/(8\alpha)\right)$ (see Figure 5).*
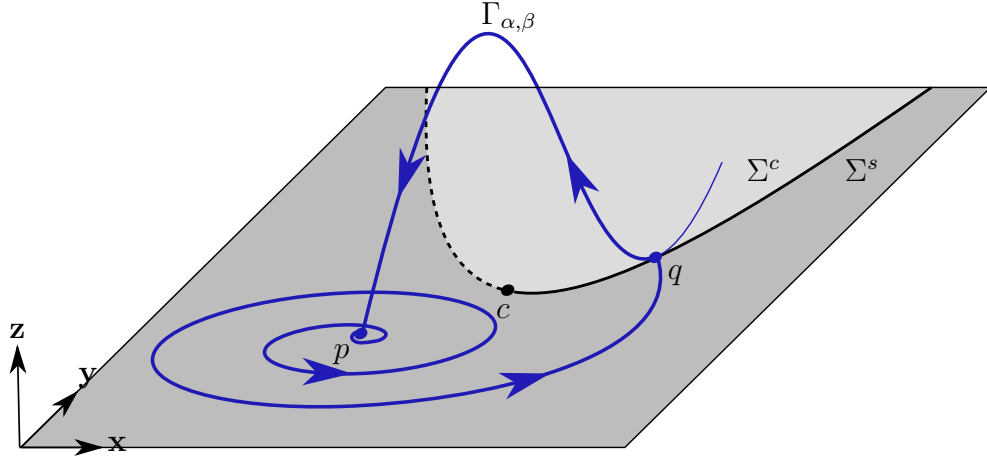

FIGURE 5. A representation of the Shilnikov sliding orbit of system (4) after bending the y–axis by doing the change of variables $(\mathbf{x}, \mathbf{y}, \mathbf{z}) = (x, y - x^2, z)$.

Proof. We compute the sliding vector field and the normalized sliding vector field of (4) as

(5)
$$\widetilde{Z}_{\alpha,\beta}(x,y) = \left(\frac{4\alpha^2 y}{4\alpha y - 3\beta^2}, \frac{3\beta^2 x + \alpha\beta^2 y - 24\alpha^2 y^2}{6\beta^3 - 8\alpha\beta y} \right) \quad \text{and}$$
$$\widehat{Z}_{\alpha,\beta}(x,y) = \left(-\alpha y, \frac{3\beta^2}{8\alpha} x + \frac{\beta}{8} y - \frac{3\alpha}{\beta} y^2 \right),$$

respectively. It is easy to see that $(0,0) \in \Sigma^s$ is a hyperbolic focus of $\widetilde{Z}_{\alpha,\beta}$. Indeed, their eigenvalues are given by

$$\lambda^\pm = \frac{\alpha}{12\beta} \pm i \frac{\sqrt{95}\alpha}{12\beta}.$$

It implies that the origin is a hyperbolic pseudo saddle–focus of vector field (4). Moreover, since $\mathrm{Re}(\lambda^{\pm}) > 0$ then $(0,0)$ is an unstable hyperbolic focus of the (normalized) sliding vector field (5).

After a change of variables and a time rescaling expressed by

$$(6) \qquad\qquad (x,y) = \left(\frac{3\beta}{2} u, \frac{3\beta^2}{8\alpha} v \right), \quad t = -\frac{4}{\beta}\tau,$$

respectively, the normalized sliding vector field $\widehat{Z}_{\alpha,\beta}$ becomes

$$(7) \qquad\qquad \overline{Z} = \left(v, -6u - \frac{1}{2}v + \frac{9}{2}v^2 \right).$$

We note that the time rescaling (6) reverses the direction of the flow of (5). The fold line $\partial\Sigma^s$ is given now, in (u,v) coordinates, by $\ell = \{(u,1) : u \in \mathbb{R}\}$.

We claim that the orbit of system (7) starting at the point $(1,1) \in \ell$ is attracted to the focus equilibrium $(0,0)$ without touching the line ℓ. Clearly, going back through the transformation (6), this claim implies that the orbit of system (5) starting at the point $q = (3\beta/2, 3\beta^2/(8\alpha)) \in \partial\Sigma^s$ is attracted, now backward in time, to the focus $(0,0)$ without touching the fold line $\partial\Sigma^s$.

To prove the claim we shall construct a compact region $\mathcal{R}$ in the u,v–plane that is positively invariant through the flow of the vector field (7). To do that, let $m(y) = -13/108 + 9y^2/13 + 54y^3/169$, and take the curves

$$\begin{aligned}
\mathcal{C}_1 &= \{(u,1) : m(1) \le y \le 1\}, \\
\mathcal{C}_2 &= \{(u, -2u+3) : 1 \le u \le 3/2\}, \\
\mathcal{C}_3 &= \{(u, -91/72) : m(-91/72) \le u \le 3/2\}, \\
\mathcal{C}_4 &= \{(3/2, v) : -91/72 < v < 0\}.
\end{aligned}$$

We define $\mathcal{R}$ as being the compact region delimited by the curves $\mathcal{C}_i$ for $i = 1, 2, \ldots, 5$ (see Figure 6). After some standard computations we conclude that $\mathcal{R}$ is positively invariant through the flow of (7). Furthermore, the vector field (7) has at most one limit cycle (see Theorem A of [4]), which is hyperbolic. So from the positive invariance of $\mathcal{R}$, from the stability of the equilibrium $(0,0)$, and from the uniqueness of a possible limit cycle we conclude that, if this limit cycles exists, then it cannot be inside $\mathcal{R}$. Applying Poincaré–Bendixson theorem we conclude that the stable focus of (7) attracts the orbits, forward in time, of all points in $\mathcal{R}$ without touching the line ℓ. The claim follows by noting that $(1,1) \in \mathcal{R}$.
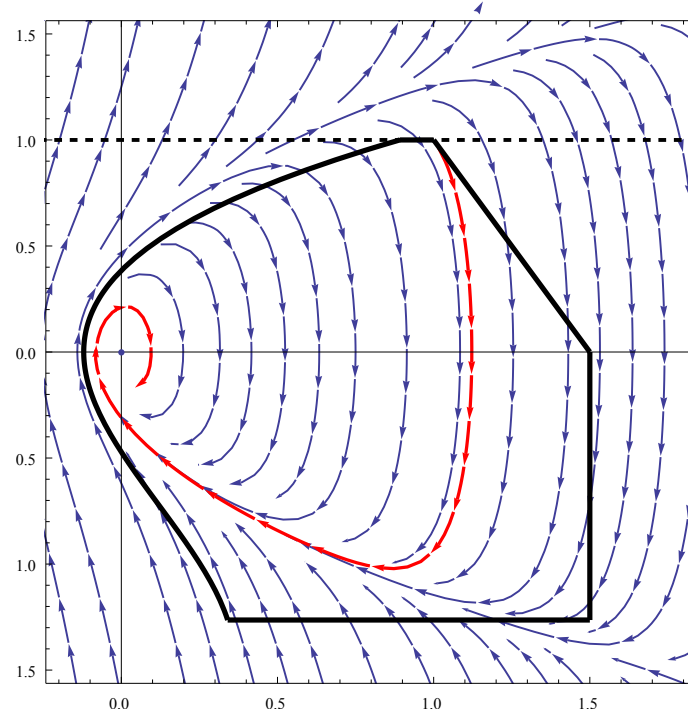
$\square$

FIGURE 6. The dashed bold line represents the line ℓ. The continuous bold line delimits the compact region $\mathcal{R}$ which is positively invariant through the flow of (7). The bold trajectory is the orbit starting at $(1,1)$ being attracted to the focus $(0,0)$.

On the other hand the vector field $X_{\alpha,\beta}$ is also linear. Thus its orbit starting at q is given by

$$\varphi^+(t,q) = \left(-\alpha\, t + \frac{3\beta}{2}\,,\, \frac{(3\beta - 2\alpha\, t)(\beta + 2\alpha\, t)}{8\alpha}\,,\, \frac{(2\beta - 2\alpha\, t)t^2}{12} \right).$$

So for $t^+ = 3\beta/(2\alpha) > 0$ we have that $\varphi^+(t^+, q) = p$. It implies that there exists a sliding Shilnikov orbit $\Gamma_{\alpha,\beta}$ of $Z_{1,\alpha,\beta}$ connecting p to itself passing through q.

3. REGULARIZATION

Shilnikov [11, 12] showed that any smooth 3–dimensional vector field possessing a hyperbolic saddle–focus $p \in \mathbb{R}^3$ with a 2–dimensional stable (unstable) manifold and an 1–dimensional unstable (stable) manifold admits a chaotic behaviour always when its saddle quantity $\sigma = \lambda^u + \mathrm{Re}(\lambda_{1,2}^s)$ (resp. $\sigma = \lambda^s + \mathrm{Re}(\lambda_{1,2}^u)$) is positive (negative). Tresser extended the Shilnikov's results for $\mathcal{C}^{1,1}$ vector fields [16] and for Lipschitz continuous piecewise $\mathcal{C}^{1,1}$

vector fields [17] when the Shilnikov homoclinic orbit is transversal to the sets of non–differentiability.

As an immediate consequence of the main result of this section we shall obtain that, for each positive real numbers α and β, every neighbourhood $\mathcal{U} \subset \Omega^0$ of the piecewise linear model $Z_{\alpha,\beta}(x, y, z)$, built in the previous section, contains a Lipschitz continuous vector field possessing a Shilnikov homoclinic orbit. Moreover this vector field presents a chaotic behaviour, and any neighbourhood of its Shilnikov homoclinic orbit contains infinitely many periodic orbits.

Theorem C. *For each positive real numbers α and β, and for $\delta > 0$ small enough, there exists a family $W_{\alpha,\beta}^{\delta}$ of continuous piecewise smooth vector fields δ–close to $Z_{\alpha,\beta}$ having the following properties for $\delta > 0$ small enough.*
   (a) The origin is a hyperbolic saddle–focus singularity of $W_{\alpha,\beta}^{\delta}$ admitting an 1–dimensional stable manifold $\mathcal{W}_{\delta}^{s}$ and a 2–dimensional unstable manifold $\mathcal{W}_{\delta}^{u}$;
   (b) The stable and unstable manifolds intersect each other in a Shilnikov homoclinic orbit $\Gamma_{\alpha,\beta}^{\delta} = \mathcal{W}_{\delta}^{s} \cap \mathcal{W}_{\delta}^{u}$, which is δ–close to $\Gamma_{a,b}$.
   (c) The saddle quantity σ of the origin is negative. So any neighbourhood of $\Gamma_{\alpha,\beta}^{\delta}$ contains infinitely many periodic orbits for every $\delta > 0$ sufficiently small.

Before proving Theorem C we describe the *regularization process*, which is the main too we shall use in its proof. Roughly speaking, a regularization of a discontinuous system $Z = (X, Y)$ is a one–parameter family Z^{δ} of continuous vector fields such that Z^{δ} converges to the discontinuous system when $\delta \to 0$. The regularized system Z^{δ} represents a class of continuous functions approximated by Z as $\delta \to 0$. The Sotomayor-Teixeira method of regularization [14] takes

$$Z^{\delta}(x, y) = \frac{1 + \phi_{\delta}(h(x, y))}{2} X(x, y) + \frac{1 - \phi_{\delta}(h(x, y))}{2} Y(x, y), \quad \text{being}$$

$$\phi_{\delta}(h) := \phi(h/\delta),$$

where $\phi : \mathbb{R} \to \mathbb{R}$ is a continuous function which is $\mathcal{C}^1$ for $s \in (-1, 1)$ such that $\phi(s) = \text{sign}(s)$ for $|s| \geq 1$, and $\phi'(s) > 0$ for $s \in (-1, 1)$. We call ϕ a *monotonic transition function* and $Z^{\delta}(x)$ the ϕ–regularization of Z.

Proof of Theorem C. Let $\phi : \mathbb{R} \to \mathbb{R}$ be the following monotonic transition function

$$\phi(u) = \begin{cases} 1 & \text{if} \quad u > 1, \\ u & \text{if} \quad -1 < u < 1, \\ -1 & \text{if} \quad u < -1. \end{cases}$$

and $h(x, y, z) = z$. We take $W_{\alpha,\beta}^{\delta} = Z_{\alpha,\beta}^{\delta} + \delta(0,\,0,\,A + Bx)$ where $Z_{\alpha,\beta}^{\delta}$ is the ϕ–regularization of the vector field (4). Thus the differential system induced by $W_{\alpha,\beta}^{\delta}$, for $-\delta \leq z \leq \delta$, is given by

$$
\begin{aligned}
\dot{x} &= -\frac{\alpha\,z}{\delta}, \\[4pt]
\dot{y} &= \frac{(\beta x - 3\alpha\,y - 2\beta^2)z}{2\delta\beta} + \frac{\beta\,x + 3\alpha\,y}{2\beta}, \\[4pt]
\dot{z} &= \frac{(4\alpha y - 3\beta^2)z}{8\delta\alpha} + \frac{y + (Ax + By)z}{2} + \delta\frac{(Ax + By)}{2}.
\end{aligned}
\tag{8}
$$

We note that for $z \geq \delta$ $W_{\alpha,\beta}^{\delta} = X_{\alpha,\beta}^{\delta} + \delta(0,\,0,\,A + Bx)$, and for $z \leq -\delta$ $W_{\alpha,\beta}^{\delta} = Y_{\alpha,\beta}^{\delta} + \delta(0,\,0,\,A + Bx)$, which are linear vector fields.

In order to simplify the study, we take $z = \delta w$. Thus system (8) becomes

$$
\begin{aligned}
\dot{x} &= -\alpha\,w, \\[4pt]
\dot{y} &= \frac{\beta\,x + 3\alpha\,y + (\beta x - 3\alpha\,y - 2\beta^2)w}{2\beta}, \\[4pt]
\delta\dot{w} &= \frac{y}{2} + \frac{(4\alpha y - 3\beta^2)w}{8\alpha} + \delta\frac{(Ax + By)(w + 1)}{2}.
\end{aligned}
\tag{9}
$$

Here dot denotes derivative with respect to the variable t. We note that the origin $(0, 0, 0)$ is the unique singularity of system (9) for every $\delta > 0$. Moreover we can estimate their eigenvalues as

$$
\lambda^s = -\frac{3\beta^2}{8\delta\alpha} + \frac{4\alpha}{3\beta} + \mathcal{O}(\delta) \quad \text{and} \quad \lambda_{1,2}^u = \frac{\alpha}{12\beta} \pm i\frac{\alpha\sqrt{95}}{12\beta} + \mathcal{O}(\delta).
$$

So we conclude that the origin is a hyperbolic saddle–focus singularity for every $\delta > 0$ sufficiently small, which has an 1–dimensional stable manifold $\mathcal{W}_{\delta}^s$ and a 2–dimensional unstable manifold $\mathcal{W}_{\delta}^u$. It concludes the proof of statement (a).

System (9), known as *slow system*, can be studied using singular perturbation methods. Doing $\delta = 0$ we obtain the *reduced problem*

$$
\begin{aligned}
\dot{x} &= -\alpha\,w, \\[4pt]
\dot{y} &= \frac{\beta\,x + 3\alpha\,y + (\beta x - 3\alpha\,y - 2\beta^2)w}{2\beta}, \\[4pt]
0 &= \frac{y}{2} + \frac{(4\alpha y - 3\beta^2)w}{8\alpha},
\end{aligned}
$$

which is a differential equation defined on a manifold. This manifold is obtained as the graph through the last equality

$$(10) \quad \mathcal{M}_0 = \left\{ (x, y, m_0(x, y)) : x \in \mathbb{R}, \ y \le \frac{3\beta^2}{8\alpha}, \ m_0(x, y) = \frac{4\alpha y}{3\beta^2 - 4\alpha y} \right\}.$$

We note that $(0, 0, 0) \in \mathcal{M}_0$.

Now performing the time rescaling $t = \delta\tau$ we get the so called *fast system*

$$(11) \qquad \begin{aligned} x' &= -\delta\alpha\, w, \\ y' &= \delta\frac{(\beta\, x + 3\alpha\, y + (\beta x - 3\alpha\, y - 2\beta^2)w)}{2\beta}, \\ w' &= \frac{y}{2} + \frac{(4\alpha y - 3\beta^2)w}{8\alpha} + \delta\frac{(Ax + By)(w + 1)}{2}. \end{aligned}$$

that we shall denote by $F_\delta(x, y, w)$. Here the prime denotes derivative with respect to the variable τ. We note that $\mathcal{M}_0$ is a manifold of critical points for system (11) when $\delta = 0$, that is

$$(12) \qquad \begin{aligned} x' &= 0, \\ y' &= 0, \\ w' &= \frac{y}{2} + \frac{(4\alpha y - 3\beta^2)w}{8\alpha}. \end{aligned}$$

System (12) is known as the *layer problem*.

Using systems (11) and (12) it is straightforward to prove that the solution $\varphi(\tau, \delta) = \big(\varphi_1(\tau, \delta), \varphi_2(\tau, \delta), \varphi_3(\tau, \delta)\big)$ of system (11) such that $f_3(0, \delta) = 1$ and $\lim_{t \to \infty} \varphi(\tau, \delta) = (0, 0, 0)$ can be estimated, for $\delta > 0$ small enough, as

$$\varphi_1(t, \delta) = \frac{8\delta\alpha^2}{3\beta^2} e^{-\frac{3\beta^2 t}{8\alpha}} + \mathcal{O}(\delta^2), \quad \varphi_2(t, \delta) = \frac{8\delta\alpha}{3\beta} e^{-\frac{3\beta^2 t}{8\alpha}} + \mathcal{O}(\delta^2), \quad \text{and}$$

$$\varphi_3(t, \delta) = e^{-\frac{3\beta^2 t}{8\alpha}} + \frac{4\delta\alpha}{9\beta^3} \left(4\alpha(8\alpha + 3\beta^2 t) e^{-\frac{3\beta^2 t}{8\alpha}} - 32 e^{-\frac{3\beta^2 t}{4\alpha}} \right) + \mathcal{O}(\delta^2).$$

So the stable manifold $\mathcal{W}_\delta^s$ intersects the plane $w = 1$ at the point

$$p_\delta = \varphi(0, \delta) = \left(\frac{\delta 8\alpha^2}{3\beta^2}, \frac{\delta 8\alpha}{3\beta}, 1 \right) + \mathcal{O}(\delta^2).$$

For $(x, y, w) \in \mathcal{M}_0$ we compute

$$DF_0(x, y, w) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \dfrac{-3\beta^2}{8\alpha y - 6\beta^2} & \dfrac{4\alpha y - 3\beta^2}{8\alpha} \end{pmatrix}$$

Since $(4\alpha y - 3\beta^2)/(8\alpha) \neq 0$ for all the points of $\mathcal{M}_0$, it follows that the manifold $\mathcal{M}_0$ is a normally hyperbolic attracting manifold for F_0. So in any compact set of $\mathcal{M}_0$ we can apply the well known *first Fenichel theorem* (see, for instance, [6, 8, 9]), which ensures the existence of a normally hyperbolic attracting invariant manifold $\mathcal{M}_\delta$ for $\delta > 0$ sufficiently small of system (9) and (11), which is known as *slow manifold*. The slow manifold $\mathcal{M}_\delta$ is δ–close to $\mathcal{M}_0$, that is $\mathcal{M}_\delta = \{(x, y, m(x, y, \delta)) : m(x, y, \delta) = m_0(x, y) + \delta m_1(x, y)\}$, where m_0 is defined in (10). Moreover we can compute m_1 as

$$m_1(x, y) = \frac{12\alpha\beta^2(A\,x + B\,y)}{(4\alpha\,y - 3\beta^2)^2} - \frac{48\alpha^2\beta(3\beta^3\,x + \alpha\beta^2\,y - 24\alpha^2 y^2)}{(4\alpha\,y - 3\beta^2)^4}.$$

We claim that the slow manifold $\mathcal{M}_\delta$ contains the origin for $\delta > 0$ sufficiently small. Indeed, suppose that $(0, 0, 0) \notin \mathcal{M}_\delta$ so it is δ–close to $\mathcal{M}_\delta$ because $(0, 0, 0) \in \mathcal{M}_0$. Since $\mathcal{M}_\delta$ is an attracting invariant manifold for $\delta > 0$ sufficiently small, it must attract the origin which is contradiction because the origin is a singularity. Thus we conclude that $(0, 0, 0) \in \mathcal{M}_\delta$ for $\delta > 0$ sufficiently small. From similar reasons the slow manifold also contains the unstable manifold $\mathcal{W}_\delta^u$ of the singularity $(0, 0, 0)$ for $\delta > 0$ sufficiently small.

We can easily check that the slow manifold $\mathcal{M}_\delta$ intersects the plane $w = 1$ transversely along the curve $(x, \ell(x, \delta), 1)$, where

$$\ell(x, \delta) = \frac{3\beta^2}{8\alpha} + \delta \left(\frac{16\alpha\,x}{3\beta^2} - Ax - \frac{3B\beta^2}{8\alpha} - \frac{16\alpha}{3\beta} \right) + \mathcal{O}(\delta^2).$$

Now we consider the solution $(x(t, \delta), y(t, \delta), w(t, \delta))$ of system (9) starting at a point of the slow manifold $\mathcal{M}_\delta$. From its invariance property we know that $w(t) = m_0(x(t, \delta), y(t, \delta)) + \delta m_1(x(t, \delta), y(t, \delta)) + \mathcal{O}(\delta^2)$. Substituting this relation in the slow system (9) we obtain the following planar differential system

$$x' = \frac{4\alpha^2 y}{4\alpha y - 3\beta^2} + \mathcal{O}(\delta),$$

$$y' = \frac{3\beta^2 x + \alpha\beta^2 y - 24\alpha^2 y^2}{6\beta^3 - 8\alpha\beta y} + \mathcal{O}(\delta),$$

which is topologically equivalent to the sliding vector field (5) for $y < 3\beta^2/(4\alpha)$ and $\delta > 0$ small enough.

Let $q_\delta = \big(3\beta/2, \ell(3\beta/2, \delta), m_\delta(q_\delta)\big)$. From the proof of Proposition 3 we know that, for $\delta = 0$, the orbit starting at $q_0 = \big(3\beta/2, 3\beta^2/(8\alpha), 1\big) = \big(q, 1\big)$ is attracted, backward in time, to the focus $(0, 0, 0)$. So, for $\delta > 0$ sufficiently small, the orbit starting at

$$q_\delta = \left(\frac{3\beta}{2}, \, \frac{3\beta^2}{8\alpha} + \frac{\delta(64\alpha^2 - 36A\,\alpha\beta^2 - 9B\,\beta^3)}{24\alpha\beta}, \, 1 \right) + \mathcal{O}(\delta^2)$$

is also attracted, backward in time, to the focus $(0, 0, 0)$.

Let $\overline{q}_\delta$ and $\overline{p}_\delta$ be the points q_δ and p_δ in the variables (x, y, z) (that is $z = \delta w$). The proof will follow by showing that for some branches A_δ and B_δ the flow of the linear system $X_{\alpha,\beta}$ connects the points $\overline{q}_\delta$ and $\overline{p}_\delta$ for $\delta > 0$ sufficiently small.

For $z \geq 1$ the vector field $W_{\alpha,\beta}^\delta$ is equal to the linear vector field $X_{\alpha,\beta}(x, y, z) + \delta\big(0, \, 0, \, A + Bx\big)$. Computing its solution $\psi(t, \delta) = \big(\psi_1(t, \delta), \psi_2(t, \delta), \psi_3(t, \delta)\big)$ such that $\psi(0, \delta) = \overline{q}_\delta$ we obtain that

$$\psi_1(t, \delta) = \frac{3\beta}{2} - \alpha\,t + \mathcal{O}(\delta^2),$$

$$\psi_2(t, \delta) = \frac{(3\beta - 2\alpha\,t)(\beta + 2\alpha\,t)}{8\alpha} + \frac{\delta(6a\alpha^2 - 36A\alpha\beta^2 - 9B\beta^3)}{24\alpha\beta} + \mathcal{O}(\delta^2),$$

$$\psi_3(t, \delta) = \frac{(3\beta - 2\alpha\,t)t^2}{12} + \frac{\delta}{12}\left(12 + \frac{32\alpha\,t}{\beta} - (6A\alpha - 3B\beta)t^2 - 2B\alpha\,t^3 \right) + \mathcal{O}(\delta^2),$$

Since the orbit $\psi(t, 0)$ reaches transversally the plane $\Sigma = \{z = 0\}$ in a finite time $t_0 = 3\beta/(2\alpha)$, we can prove that the orbit $\psi(t, \delta)$, for $\delta > 0$ small enough, will also reach transversally the plane $z = \delta$ in a finite time t_δ. Moreover we can estimate $t_\delta = 3\beta/(2\alpha) + \delta(32\alpha - 9A\beta^2)/(3\beta^2) + \mathcal{O}(\delta^2)$.

Let $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ and $\pi^\perp : \mathbb{R}^3 \to \mathbb{R}$ be the projections onto the two first coordinates and onto the last coordinate, respectively. Define $\mathcal{F}(A, B, \delta) = (\psi(t_\delta, \delta) - \overline{p}_\delta)/\delta$. It is easy to see that, for every $\delta > 0$ sufficiently small, $\pi^\perp\mathcal{F}(A, B, \delta) = 0$ and

$$\pi\mathcal{F}(A, B, \delta) = \left(3A\alpha - \frac{40\alpha^2}{3\beta^2} + \mathcal{O}(\delta), \, -\frac{32\alpha}{3\beta} + \frac{3A\beta}{2} - \frac{3B\beta^2}{8\alpha} + \mathcal{O}(\delta) \right).$$

We note that $\mathcal{F}(A_0, B_0, \delta_0) = 0$, for some A_0, B_0, and $\delta_0 > 0$, if and only if the vector field (8) (for $A = A_0$, $B = B_0$, and $\delta = \delta_0$) admits an orbit connecting the points $\overline{q}_{\delta_0}$ and $\overline{p}_{\delta_0}$, that is an sliding Shilnikov orbit. Since for $A^* = 40\alpha/(9\beta^2)$ and $B^* = -32\alpha^2/(3\beta^3)$, we have that $\pi\mathcal{F}(A^*, B^*, 0) = 0$ and $\det(\pi D\mathcal{F}(A^*, B^*, 0)) = -9\beta^2/8 \neq 0$, then, using the *implicit function Theorem*, we conclude that, for $\delta > 0$ sufficiently small, there exist two branches A_δ and B_δ such that $\pi\mathcal{F}(A_\delta, B_\delta, \delta) = 0$, and $A_\delta \to A^*$ and $B_\delta \to B^*$ when $\delta \to 0$. It concludes the proof of statement (b).

Finally, we compute the saddle quantity as $\sigma = -3\beta^2/(8\delta\,\alpha) + 17\alpha/(12\beta) + \mathcal{O}(\delta)$ which negative for $\delta > 0$ small enough. The proof of statement (c) follows by applying the classical results for Shilnikov homoclinic orbits [16, 17].

<div style="text-align: right;">□</div>

4. Conclusions and further directions

In this paper we have given versions of the Shilnikov's Theorems for systems having a sliding Shilnikov orbit, which is a type of connection (introduced in this work) in the context of nonsmooth dynamical systems. We have also built, explicitly, a family $Z_{\alpha,\beta}$, of discontinuous piecewise linear vector fields, having this kind of connection. Studying its regularization we provided a bridge between the classical theory and its analogous in nonsmooth dynamical theory.

Higher dimension vector fields allows the existence of many other kinds of sliding homoclinic connections. So the study of homoclinic sliding orbits in higher dimension is a natural direction for further investigations.

Finally, let Z be a piecewise vector field in Ω^r admitting a sliding Shilnikov orbit Γ. We conjecture that, for all $s \geq r$, every neighbourhood $\mathcal{U} \subset \Omega^r$ of Z contains a vector field $W \in \mathcal{C}^s$ possessing a Shilnikov homoclinic orbit $\widetilde{\Gamma}$.

Acknowledgements

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